Calculation of variance of loss in restricted energy range

Macroscopic cross-section without spin term:

$$\frac{d\Sigma}{dT} = \frac{K}{T^2}\left(1 - \beta^2 \frac{T}{T_{max}}\right),$$

$$K = 2\pi r_e^2 m_e c^2 n_{el} \frac{z^2}{\beta^2}.$$

Macroscopic cross-section:

$$\Sigma = \int \frac{d\Sigma}{dT} dT.$$

Probability density function for an event:

$$\frac{1}{\Sigma}\frac{d\Sigma}{dT}.$$

Average loss per event:

$$E(T) = \frac{1}{\Sigma}\int \frac{d\Sigma}{dT} T dT.$$

Average square of loss per event:

$$E(T^2) = \frac{1}{\Sigma}\int \frac{d\Sigma}{dT} T^2 dT.$$

Variance of loss per event:

$$Var(T) = E(T^2) - E(T)^2.$$

Average number of events in distance l:

$$\lambda = \Sigma l.$$

Number of events is Poisson distributed with parameter λ, probability of N events is:

$$P(N;\lambda) = e^{-\lambda}\frac{\lambda^N}{N!}.$$

Average N:

$$E(N) = \lambda.$$

Variance of N:

$$Var(N) = \lambda.$$

Sum of losses:

$$\Delta = \sum_{i=1}^{i=N} T_i.$$

Average loss in case of N events:

$$E(\Delta|N) = NE(T).$$

Variance in case of N events:

$$Var(\Delta|N) = NVar(T).$$

Average square of loss in case of N events:

$$E(\Delta^2|N) = Var(\Delta|N) + E(\Delta|N)^2 = NVar(T) + N^2E(T)^2.$$

Average loss:

$$E(\Delta) = \sum_{N=0}^{\infty} E\,(\Delta|N)P(N;\lambda) = \sum_{N=0}^{\infty} N\,E(T)P(N;\lambda) = E(N)E(T).$$

Average square of loss:

$$E(\Delta^2) = \sum_{N=0}^{\infty} E\,(\Delta^2|N)P(N;\lambda) = \sum_{N=0}^{\infty} (NVar(T) + N^2E(T)^2)\,P(N;\lambda)$$
$$= E(N)Var(T) + E(N^2)E(T)^2.$$

Variance of loss:

$$Var(\Delta) = E(\Delta^2) - E(\Delta)^2 = E(N)Var(T) + E(N^2)E(T)^2 - E(N)^2E(T)^2,$$

$$Var(\Delta) = E(N)Var(T) + Var(N)E(T)^2 = \lambda Var(T) + \lambda E(T)^2 = \lambda E(T^2).$$

$$Var(\Delta) = \Sigma l \frac{1}{\Sigma} \int \frac{d\Sigma}{dT} T^2 dT = l \int \frac{d\Sigma}{dT} T^2 dT = Kl \int \left(1 - \beta^2 \frac{T}{T_{max}}\right) dT.$$

$$Var(\Delta) = Kl\left(T_{cut} - \beta^2 \frac{{T_{cut}}^2}{2T_{max}}\right) = KlT\left(1 - \frac{\beta^2}{2}\frac{T_{cut}}{T_{max}}\right).$$

$$Var(\Delta) \neq KlT_{cut}\left(1 - \frac{\beta^2}{2}\right),$$